Calculus 2018 Hustle Solutions

1. $\lim\limits_{x\to 0}\left[\frac{\sin 4x}{x}\right]=\lim\limits_{x\to 0}\left[\frac{\sin 4x}{x}\right]\cdot\frac{4}{4}=4\lim\limits_{x\to 0}\left[\frac{\sin 4x}{4x}\right]=4\cdot 1=4$

2. let $x=\frac{1}{y}\quad y=\frac{1}{x}\quad \lim\limits_{x\to 0} y=\infty$

$$\lim_{x\to 0}\left[\frac{\ln(1+x)}{x}\right]=\lim_{x\to 0}\left[\frac{1}{x}\cdot\ln(1+x)\right]=\lim_{x\to 0}\left[\ln(1+x)^{1/x}\right]=\lim_{y\to\infty}\left[\ln\left(1+\frac{1}{y}\right)^{y}\right]=\ln e=1$$

3. $y=\frac{xe^x-x^2}{x+e^x}$

$$\begin{aligned}\frac{dy}{dx}&=\frac{(x+e^x)\left[(xe^x+e^x)-2x\right]-(xe^x-x^2)(1+e^x)}{(x+e^x)^2}\\&=\frac{x^2e^x+xe^x-2x^2+xe^{2x}+e^{2x}-2xe^x-xe^x-xe^{2x}+x^2+x^2e^x}{(x+e^x)^2}\\&=\frac{2x^2e^x-x^2-2xe^x+e^{2x}}{(x+e^x)^2}\end{aligned}$$

4. let $y=\ln u$, $u=\tan x$

$$\frac{dy}{du}=\frac{1}{u}\qquad \frac{du}{dx}=\sec^2 x=\frac{1}{\cos^2 x}\qquad \frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\frac{1}{u}\cdot\frac{1}{\cos^2 x}=\frac{1}{\tan x\cos^2 x}=\frac{1}{\sin x\cos x}=\frac{2}{\sin 2x}$$

5. Let $\frac{dy}{dt}=-\frac{dx}{dt}$

$$16x^2+9y^2=400\Rightarrow 32x\frac{dx}{dt}+18y\frac{dy}{dt}=0\Rightarrow 32x\frac{dx}{dt}-18y\frac{dx}{dt}=0\Rightarrow 32x=18y\Rightarrow y=\frac{32}{18}x=\frac{16}{9}x$$

$$16x^2+9\left(\frac{16}{9}x\right)^2=400\Rightarrow 16x^2+\frac{256}{9}x^2=400\Rightarrow\frac{400}{9}x^2=400\Rightarrow x=\pm 3$$

$$\left(3,\frac{16}{3}\right),\left(-3,\frac{-16}{3}\right)$$

6. $c = 15$ cm Looking for $\frac{dd}{dt}$ when $\frac{dc}{dt} = 0.2\,\text{cm/sec}$

$$\frac{dd}{dt} = \frac{(c-10)\cdot 10\cdot \frac{dc}{dt} - 10c\cdot \frac{dc}{dt}}{(c-10)^2} = \frac{(15-10)\cdot 10\cdot (0.2) - 10(15)(0.2)}{(15-10)^2} = -0.8\,\text{cm/sec}$$

7. $y = x\ln x \Rightarrow y' = x\cdot \frac{1}{x} + \ln x = 1 + \ln x$ $2x - 2y + 3 = 0 \rightarrow m = 1$

$\frac{-1}{1+\ln x_0} = 1 \Rightarrow -1 = 1 + \ln x_0 \Rightarrow 2 = \ln x_0 \Rightarrow x_0 = \frac{1}{e^2}$. Therefore, $y_0 = \frac{1}{e^2}\cdot \ln e^{-2} = \frac{-2}{e^2}$

$$y + \frac{2}{e^2} = 1\left(x - \frac{1}{e^2}\right)$$

8. Let $u = b + ax \Rightarrow du = adx$

$$\int (b+ax)^{-2/3}dx = \frac{1}{a}\int a(b+ax)^{-2/3}\,dx = \frac{1}{a}\int u^{-2/3}du = \frac{1}{a}\cdot u^{1/3}\cdot \frac{3}{1} + C = \frac{3}{a}(b+ax)^{1/3} + C$$

9. $$\int \frac{1+2x^2}{x^2(1+x^2)}dx = \int \frac{1+x^2}{x^2(1+x^2)}dx + \int \frac{x^2}{x^2(1+x^2)}dx = \int \frac{dx}{x^2} + \int \frac{dx}{1+x^2} = \frac{-1}{x} + \arctan x + C$$

10. By polynomial long division, $\frac{5x^2+2x-3}{x+2} = 5x - 8 + \frac{13}{x+2}$

$$\int \frac{5x^2+2x-3}{x+2}\,dx = \int (5x-8)\,dx + 13\int \frac{dx}{x+2} = \frac{5x^2}{2} - 8x + 13\ln|x+2| + C$$

11. $$\frac{d}{dx}\int_5^{x^2} \sqrt{t^2+3t+1}\,dt = \sqrt{\left(x^2\right)^2 + 3\left(x^2\right) + 1}\cdot 2x = 2x\sqrt{x^4+3x^2+1}$$

12. $u = e^x \qquad du = e^x dx$

$$v = \sin x \qquad dv = \cos x dx$$
$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx$$
$$u = e^x \qquad du = e^x dx$$
$$v = -\cos x \qquad dv = \sin x dx$$
$$\int e^x \cos x dx = e^x \sin x - \left[-e^x \cos x + \int e^x \cos x dx \right]$$
$$\int e^x \cos x dx = e^x \sin x + e^x \cos x - \int e^x \cos x dx$$
$$2\int e^x \cos x dx = e^x \sin x + e^x \cos x$$
$$\int e^x \cos x dx = \frac{e^x}{2}(\sin x + \cos x) + C$$

13. $\dfrac{2x-34}{3x^2-11x-4} = \dfrac{8}{3x+1} - \dfrac{2}{x-4}$ by partial fraction decomposition.

$$\int \frac{2x-34}{3x^2-11x-4} dx = \int \frac{8}{3x+1} dx - \int \frac{2}{x-4} dx = \frac{8}{3}\int \frac{3}{3x+1} dx - 2\int \frac{1}{x-4} dx = \frac{8}{3}\ln|3x+1| - 2\ln|x-4| + C$$

14. $\displaystyle\int_{-a}^{a} \frac{dx}{\sqrt{a^2-x^2}} = \arcsin \frac{x}{a}\Big|_{-a}^{a} = \arcsin 1 - \arcsin(-1) = \frac{\pi}{2} - \frac{-\pi}{2} = \pi$

15. If $(2,5)$ is on *f*, then $(5,2)$ is on *g*. $g'(5) = \dfrac{1}{f'(g(5))} = \dfrac{1}{f'(2)} = 1$

16. $y' = \dfrac{\sqrt{x^2+1}(2x) - x^2\left[\frac{1}{2}(x^2+1)^{-1/2}(2x)\right]}{x^2+1}$

$$2x\sqrt{x^2+1} - \frac{x^3}{\sqrt{x^2+1}} = 0$$
$$\frac{2x(x^2+1) - x^3}{\sqrt{x^2+1}} = 0$$
$$2x^3 + 2x - x^3 = 0$$
$$x^3 + 2x = 0$$
$$x(x^2+2) = 0$$
$$x = 0$$

17. $\frac{dy}{dx} = 3 - 3x^2 \Rightarrow$ slope of tangent line at $x = 3$ is $3 - 3(3)^2 = -24$

slope of normal line at $x = 3$ is $\frac{1}{24}$.

$$y + 18 = \frac{1}{24}(x - 3) \Rightarrow y - \text{intercept} = \frac{-145}{8}$$

18. $I = \frac{k}{x^2}$ $\lim_{x \to 0} \frac{k}{x^2} = \infty$

19. let w = width and l = length $w^2 + d^2 = 16^2$, feasible domain: [0,16], k = a constant

$$S = kwd^2$$
$$S = kw(256 - w^2)$$
$$S = 256kw - kw^3$$
$$S' = 256k - 3kw^2$$
$$256k - 3kw^2 = 0$$
$$w = \sqrt{\frac{256}{3}}\text{in} \quad d = \sqrt{\frac{512}{3}}\text{in}$$

20.

$$x - x^2 = ax$$
$$x^2 - x + ax = 0$$
$$x(x + a - 1) = 0$$
$$x = 0, \quad x = 1 - a$$

$$\int_0^{1-a} (x - x^2 - ax)\,dx = \frac{9}{2}$$
$$= \left(\frac{x^2}{2} - \frac{x^3}{3} - \frac{ax^2}{2}\right)\Bigg|_0^{1-a} = \frac{9}{2}$$
$$6 \cdot \left[\frac{(1-a)^2}{2} - \frac{(1-a)^3}{3} - \frac{a(1-a)^2}{2}\right] = \frac{9}{2} \cdot 6$$
$$3(1 - 2a + a^2) - 2(1 - 3a + 3a^2 - a^3) - 3a(1 - 2a + a^2) = 27$$
$$-a^3 + 3a^2 - 3a + 1 = 27$$
$$(1-a)^3 = 27$$
$$1 - a = 3$$
$$a = -2$$

21. let $u=\sin x \quad u'=\cos x$

$$\frac{d}{dx}\sin u=\cos u\cdot u'$$

$$f(x)=\sin[\sin u]$$

$$v=\sin u \quad v'=\cos u\cdot u'$$

$$f(x)=\sin v$$

$$f'(x)=\cos v\cdot v'=\cos(\sin(\sin x))\cdot\cos(\sin x)\cdot\cos x$$

22. $$f'(x)=\frac{\sin x^2\cdot[2\sin x\cos x]-\sin^2 x\cdot\left[\cos x^2\cdot 2x\right]}{\left(\sin x^2\right)^2}=\frac{2\sin x\left[\sin x^2\cos x-x\sin x\cos x^2\right]}{\sin^2 x^2}$$

23. $$\frac{1}{4-1}\int_1^4\left(2x^2+3x+3\right)dx=\frac{1}{3}\left[\left(\frac{2x^3}{3}+\frac{3x^2}{2}+3x\right)\Bigg|_1^4\right]=\frac{49}{2}$$

24. $$x=\frac{y^2}{8}$$

$$8y=\left(\frac{y^2}{8}\right)^2$$

$$\frac{y^4}{64}-8y=0$$

$$y\left(y^3-512\right)=0$$

$$y=8,\quad x=8$$

$$y=0,\quad x=0$$

$$v=\pi\int_0^8\left[\left(\sqrt{8x}\right)^2-\left(\frac{x^2}{8}\right)^2\right]dx$$

$$=\pi\int_0^8\left(8x-\frac{x^4}{64}\right)dx$$

$$=\pi\left[\left(4x^2-\frac{x^5}{320}\right)\Bigg|_0^8\right]=\frac{768\pi}{5}$$

25. $\frac{dy}{dx} = \frac{5x+2}{7y}$

$$\int 7y\,dy = \int (5x+2)\,dx$$
$$\frac{7y^2}{2} = \frac{5x^2}{2} + 2x + C$$
$$\frac{7}{2} = \frac{5\cdot 2^2}{2} + 2\cdot 2 + C$$
$$C = \frac{-21}{2}$$

$$\frac{7y^2}{2} = \frac{5x^2}{2} + 2x - \frac{21}{2}$$
$$y = \sqrt{\frac{5x^2+4x-21}{7}} \text{ (based on initial condition)}$$

Calculus 2018 Hustle answers

1. 4

2. 1

3. $\frac{2x^2e^x - x^2 - 2xe^x + e^{2x}}{\left(x+e^x\right)^2}$

4. $\frac{1}{\sin x \cos x} = \frac{2}{\sin 2x}$ or equivalent

5. $\left(3, \frac{16}{3}\right), \left(-3, \frac{-16}{3}\right)$

6. $-0.8\,\text{cm/sec}$

7. $y + \frac{2}{e^2} = 1\left(x - \frac{1}{e^2}\right)$ or equivalent

8. $\frac{3}{a}(b+ax)^{1/3} + C$ or equivalent

9. $\frac{-1}{x} + \arctan x + C$ or equivalent

10. $\frac{5x^2}{2} - 8x + 13\ln|x+2| + C$ or equivalent

11. $2x\sqrt{x^4+3x^2+1}$

12. $\frac{e^x}{2}(\sin x+\cos x)+C$ or equivalent

13. $\frac{8}{3}\ln|3x+1|-2\ln|x-4|+C$ or equivalent

14. π

15. 1

16. $x=0$

17. $\frac{-145}{8}$

18. ∞

19. $w=\frac{16\sqrt{3}}{3}\text{ in}$ $\quad d=\frac{16\sqrt{6}}{3}\text{ in}$

20. $a=-2$

21. $f'(x)=\cos(\sin(\sin x))\cdot\cos(\sin x)\cdot\cos x$

22. $f'(x)=\frac{2\sin x\left[\sin x^2\cos x-x\sin x\cos x^2\right]}{\sin^2 x^2}$

23. $\frac{49}{2}$

24. $\frac{768\pi}{5}$

25. $y=\sqrt{\frac{5x^2+4x-21}{7}}$